

## STRONG RING EXTENSIONS AND $\phi$ -PSEUDO-VALUATION RINGS

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ABSTRACT. In this paper, we extend the concept of strong extensions of domains to the context of (commutative) rings with zero-divisors. Let  $T$  be an extension ring of a ring  $R$ . A prime ideal  $P$  of  $R$  is called  $T$ -strong if, whenever  $x, y \in T$  satisfy  $xy \in P$ , then either  $x \in P$  or  $y \in P$ . If each  $P \in \text{Spec}(R)$  is  $T$ -strong, we say that  $R \subseteq T$  is a *strong extension*. We use the concept of strong extension to give a characterization of  $\phi$ -pseudo-valuation rings. We show that the theory of strong extensions of rings resembles that of strong extensions of domains.

### 1. INTRODUCTION

We assume throughout that all rings are commutative with  $1 \neq 0$  and that all ring extensions are unital. If  $R$  is a ring, then  $\text{Tot}(R)$  denotes the total quotient ring of  $R$ ,  $Z(R)$  denotes the set of zero-divisors of  $R$ ,  $\text{Nil}(R)$  denotes the set of nilpotent elements of  $R$ ,  $U(R)$  denotes the set of units of  $R$ ,  $\text{Spec}(R)$  denotes the set of prime ideals of  $R$ , and  $R_{\text{red}} := R/\text{Nil}(R)$  denotes the canonically associated reduced ring of  $R$ . "Dimension" and " $\dim(\dots)$ " refer to Krull dimension. We devote the next four paragraphs to recalling some background material.

As in [15], let  $R \subseteq T$  be an extension of (integral) domains. A prime ideal  $P$  of  $R$  is called  $T$ -strong if, whenever  $x, y \in T$  satisfy  $xy \in P$ , then either  $x \in P$  or  $y \in P$ . If each  $P \in \text{Spec}(R)$  is  $T$ -strong, we say that  $R \subseteq T$  is a *strong extension*. The concept of strong extension of domains is a generalization of the study of pseudo-valuation domains, a type of quasilocal integral domain introduced by Hedstrom-Houston in [16]. Recall that a domain  $R$ , with quotient field  $K$ , is said to be a *pseudo-valuation domain* (or, in short, a PVD) if each  $P \in \text{Spec}(R)$

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satisfies the following condition: whenever  $x, y \in K$  and  $xy \in P$ , then either  $x \in P$  or  $y \in P$ . Thus, the concept of strong extensions of domains broadens the study of the above condition by allowing the role of  $K$  to be played by an arbitrary domain containing  $R$ .

In [9], D. F. Anderson and the authors generalized the concept of pseudo-valuation domains to the context of arbitrary rings (rings  $R$  with  $Z(R)$  possibly nonzero). Recall from [9] that a ring  $R$  is called a *pseudo-valuation ring (PVR)* if every prime ideal of  $R$  is *strongly prime*, in the sense that  $aP$  and  $bR$  are comparable under inclusion for all  $a, b \in R$ . It was shown in [9] that a domain is a PVD if and only if it is a PVR.

In [3], the first-named author gave another generalization of pseudo-valuation rings. Recall from [12] and [8] that a prime ideal of a ring  $R$  is called a *divided prime (ideal of  $R$ )* if  $P \subseteq (x)$  for each  $x \in R \setminus P$ . Thus, a divided prime ideal of  $R$  is comparable under inclusion with every ideal of  $R$ . Let  $\mathcal{H} := \{R \mid R \text{ is a ring and } Nil(R) \text{ is a divided prime ideal of } R\}$ . Observe that if  $R$  is a domain, then  $R \in \mathcal{H}$ . For any ring  $R \in \mathcal{H}$ , the ring homomorphism  $\phi = \phi_R : Tot(R) \rightarrow R_{Nil(R)}$ , given by  $\phi(a/b) := a/b$  for each  $a \in R$  and  $b \in R \setminus Z(R)$ , was introduced in [3]. Note that  $\phi|_R : R \rightarrow R_{Nil(R)}$  is a ring homomorphism satisfying  $\phi(x) = x/1$  for each  $x \in R$ ; and that  $Tot(\phi(R)) = R_{Nil(R)}$ . (Note also that the proofs of these two assertions do not use the “divided” aspect of the prime ideal  $Nil(R)$  of  $R$ .)

Let  $R \in \mathcal{H}$ , and put  $K := R_{Nil(R)}$ . As in [3], a prime ideal  $Q$  of  $\phi(R)$  is said to be  *$K$ -strongly prime* if, whenever  $x, y \in K$  and  $xy \in Q$ , then either  $x \in Q$  or  $y \in Q$ . A prime ideal  $P$  of  $R$  is said to be a  *$\phi$ -strongly prime ideal of  $R$*  if  $\phi(P)$  is a  $K$ -strongly prime ideal of  $\phi(R)$ . It is known that the prime ideals of  $\phi(R)$  are the sets that are (uniquely) expressible as  $\phi(P)$  for some prime ideal  $P$  of  $R$  (cf. [1, Lemma 2.5]), the key fact being that  $\ker(\phi) \subseteq Nil(R)$ . If each  $P \in Spec(R)$  is a  $\phi$ -strongly prime ideal, then  $R$  is called a  *$\phi$ -pseudo-valuation ring ( $\phi$ -PVR)*. It was shown in [6, Proposition 2.9] that a ring  $R \in \mathcal{H}$  is a  $\phi$ -PVR if and only if  $R_{red}$  is a PVD. Any PVR in  $\mathcal{H}$  is a  $\phi$ -PVR. An example of a  $\phi$ -PVR which is not a PVR was given in [4].

Once again, let  $R \in \mathcal{H}$ , with  $K := R_{Nil(R)}$ . Recall from [5] that  $R$  is called a  *$\phi$ -chained ring* if, for every  $x \in K \setminus \phi(R)$ , we have  $x^{-1} \in \phi(R)$ . It is known that a ring  $R \in \mathcal{H}$  is a  $\phi$ -chained ring if and only if  $R_{red}$  is a valuation domain [1, Theorem 2.7]. Observe that since  $Nil(R)$  is a divided prime ideal of  $R$ , then  $Nil(R)$  is also the nilradical of  $Tot(R)$  and that  $\ker(\phi)$  is a common ideal of  $R$  and  $Tot(R)$ . Other useful features of each ring  $R \in \mathcal{H}$  include the following: (i)  $\phi(R) \in \mathcal{H}$ ; (ii)  $Tot(\phi(R)) = R_{Nil(R)}$  has only one prime ideal, namely,  $Nil(\phi(R))$ ; (iii)  $\phi(R)$  is naturally isomorphic to  $R/Ker(\phi)$ ; (iv)  $Z(\phi(R)) = Nil(\phi(R)) = \phi(Nil(R)) =$

$Nil(R_{Nil(R)})$ ; and (v)  $(R_{Nil(R)})_{red} = R_{Nil(R)}/Nil(\phi(R)) = Tot(\phi(R))/Nil(\phi(R))$  is the quotient field of  $\phi(R)_{red}$ . If  $I$  is a nonnil ideal of a ring  $R \in \mathcal{H}$ , observe that  $Nil(R) \subseteq I$ . For further studies on rings that are in the class  $\mathcal{H}$ , we recommend [1], [2], [3], [4], [5], [6], [7], and [11].

In this paper, we extend the concept of strong extensions of domains to the context of rings with zero-divisors. Let  $T$  be an extension ring of a ring  $R$ . A prime ideal  $P$  of  $R$  is called *T-strong* if, whenever  $x, y \in T$  satisfy  $xy \in P$ , then either  $x \in P$  or  $y \in P$ . If each  $P \in Spec(R)$  is *T-strong*, we say that  $R \subseteq T$  is a *strong extension*. We use the concept of strong extension of rings to characterize  $\phi$ -pseudo-valuation rings (see Corollaries 2.10 and 2.11) as well as several other classes of rings. In short, we show that the theory of strong extensions of rings resembles that of strong extensions of domains by establishing the  $\phi$ -theoretic analogues of many of the results in [15].

Throughout the paper, we use the technique of idealization of a module to construct examples. Recall that for an  $R$ -module  $B$ , the *idealization of  $B$  over  $R$*  is the ring formed from  $R \times B$  by defining addition and multiplication as  $(r, a) + (s, b) := (r + s, a + b)$  and  $(r, a)(s, b) := (rs, rb + sa)$ , respectively. A standard notation for this “idealized ring” is  $R(+B)$ . See [17] for basic properties of rings resulting from the idealization construction.

## 2. STRONG OVERRING EXTENSIONS

We begin by considering the passage of the “strong extension” property from a ring extension to the induced extension of the associated reduced rings.

**Theorem 2.1.** *Let  $R \subseteq T$  be rings. Then the following conditions are equivalent:*

- (1)  $Nil(R) = Nil(T)$  and  $R_{red} \subseteq T_{red}$  is a strong extension;
- (2)  $R \subseteq T$  is a strong extension.

PROOF. (2)  $\Rightarrow$  (1): Assume (2). Of course,  $Nil(R) \subseteq Nil(T)$ . To prove the reverse inclusion, it suffices to show that if  $w \in Nil(T)$ , then  $w \in P$  for each  $P \in Spec(R)$ . This, in turn, follows from the facts that  $w^n = 0 \in P$  for some  $n \geq 1$  and  $P$  is *T-strong*. Therefore,  $Nil(R) = Nil(T)$ .

It remains to prove that  $A := R_{red} \subseteq B := T_{red}$  is a strong extension. Note, by what we just proved, that  $B = T/Nil(R)$ . Our task is to show that if  $\bar{P} \in Spec(A)$  and  $\bar{x}, \bar{y} \in B$  satisfy  $\bar{x} \cdot \bar{y} \in \bar{P}$ , then either  $\bar{x} \in \bar{P}$  or  $\bar{y} \in \bar{P}$ . Write  $\bar{P} = P + Nil(R)$ ,  $\bar{x} = x + Nil(R)$  and  $\bar{y} = y + Nil(R)$ , for some  $P \in Spec(R)$  and  $x, y \in T$ . As  $\bar{x} \cdot \bar{y} \in \bar{P}$ , it follows that  $xy \in P + Nil(R) = P$ . Therefore, by (2), either  $x \in P$  or  $y \in P$ , and so either  $\bar{x} \in \bar{P}$  or  $\bar{y} \in \bar{P}$ , as required.

(1)  $\Rightarrow$  (2): Assume (1). We must show that if  $P \in Spec(R)$  and  $x, y \in T$  satisfy

$xy \in P$ , then either  $x \in P$  or  $y \in P$ . Consider  $A := R_{red} \subseteq B := T/Nil(T)$ , which, by (1), is a strong extension. Also, by (1),  $B = T/Nil(R)$ . Consider  $\bar{P} := P + Nil(R) \in \text{Spec}(A)$  and  $\bar{x} := x + Nil(R)$ ,  $\bar{y} := y + Nil(R) \in B$ . As  $\bar{x} \cdot \bar{y} = xy + Nil(R) \in P + Nil(R) = \bar{P}$  and  $\bar{P}$  is  $T$ -strong, it follows that either  $\bar{x} \in \bar{P}$  or  $\bar{y} \in \bar{P}$ . Thus, either  $x \in P$  or  $y \in P$ , to complete the proof.  $\square$

Example 2.3 will establish that the conditions in (1) in Theorem 2.1 are logically independent. That discussion includes a connection with the next result, which is of independent interest.

**Proposition 2.2.** *Let  $R \subseteq T$  be a strong extension of rings. Then the radical of  $(R : T)$  in  $T$  coincides with the radical of  $(R : T)$  in  $R$ ; that is,  $\{t \in T \mid t^n \in (R : T) \text{ for some positive integer } n\} = \{r \in R \mid r^n \in (R : T) \text{ for some positive integer } n\}$ .*

PROOF. Without loss of generality,  $R \neq T$ . Next, note that one inclusion is trivial. Therefore, it suffices to show that if  $t \in T$  is such that  $u := t^n \in (R : T)$  for some positive integer  $n$ , then  $t \in R$ . If  $u \in U(R)$ , then it follows from the fact that  $uT \subseteq R$  that  $T = u^{-1}uT \subseteq u^{-1}R = R$ , a contradiction. Thus,  $u$  is a nonunit of  $R$ , and so we can choose  $P \in \text{Spec}(R)$  such that  $t^n = u \in P$ . As  $P$  is  $T$ -strong,  $t \in P \subseteq R$ , to complete the proof.  $\square$

**Example 2.3.** *For ring extensions  $R \subseteq T$ , the conditions “ $Nil(R) = Nil(T)$ ” and “ $R_{red} \subseteq T_{red}$  is a strong extension” are logically independent.*

PROOF. Perhaps the easiest way to see that the first condition does not imply the second condition is to let  $R$  be any domain that is not a PVD (for instance,  $\mathbf{Z}$  or  $\mathbf{Q}[[X, Y]]$ ) and then take  $T$  to be the quotient field of  $R$ . In this example,  $Nil(R) = 0 = Nil(T)$ , although  $R_{red} \cong R \subseteq T \cong T_{red}$  is not a strong extension (since  $R$  is not a PVD).

One way to see that the second condition does not imply the first condition is the following. Let  $R$  be any (nonzero) ring, and then let  $T$  denote the ring of dual numbers over  $R$ ; that is,  $T := R[X]/(X^2) = R \oplus Rx$ , where  $x := X + (X^2)$  satisfies  $x^2 = 0$ . If  $a, b \in R$  and  $n \geq 1$ , it is easy to prove by induction that  $(a + bx)^n = a^n + na^{n-1}bx$ . As a consequence,  $Nil(T) = Nil(R) \oplus Rx$ , which is unequal to  $Nil(R)$  since  $R$  is nonzero. On the other hand,  $R_{red} \subseteq T_{red}$  is a strong extension for the most trivial of reasons. Indeed, the above description of  $Nil(T)$  easily leads to the canonical map  $R_{red} \rightarrow T_{red}$  being an isomorphism.  $\square$

It is interesting to note that the preceding example  $R \subset T$  complements Theorem 2.1 and Proposition 2.2. Indeed, in that example,  $(R : T) = 0$ , and so the

presence in  $T \setminus R$  of the nilpotent element  $x$  ensures, by virtue of Proposition 2.2, that  $R \subset T$  is not a strong extension. Accordingly, by Theorem 2.1, either  $Nil(R)$  fails to coincide with  $Nil(T)$  or  $R_{red} \subset T_{red}$  fails to be a strong extension. Of course, the above discussion shows that that exactly one of these failures occurs.

The next example shows that strong extensions with nontrivial zero-divisors are plentiful.

**Example 2.4.** *There exists a strong overring extension  $R \subset T$  (of distinct rings) for which neither  $R$  nor  $T$  is a domain and neither  $R$  nor  $T$  is quasilocal.*

PROOF. By the proof of [15, Example 2.1], there exists a strong overring extension of distinct domains  $A \subset B$  for which neither  $A$  nor  $B$  is quasilocal. Consider the idealizations  $R := A(+ )B$  and  $T := B(+ )B$ . Then it is routine to verify that  $Nil(R) = Nil(T) = Z(R) = Z(T) = \{0\}(+ )B$ . In particular, neither  $R$  nor  $T$  is a domain. Moreover,  $R$  (resp,  $T$ ) is not quasilocal since  $A$  (resp.,  $B$ ) is not quasilocal. A calculation reveals that  $T$  is an overring of  $R$ . (In detail, if  $b \in B$  is expressed as  $b = a_1 a_2^{-1}$  for suitable  $a_1, a_2 \in A$ , then  $(b, 0)(a_2, 0) = (a_1, 0) \in T$ , whence  $(b, 0) = (a_1, 0)(a_2, 0)^{-1}$  in the total quotient ring of  $T$ .) Finally, since  $R_{red} \cong A \subset B \cong T_{red}$  is a strong extension, it follows from Theorem 2.1 that  $R \subset T$  is a strong extension. The proof is complete.  $\square$

Beginning with a characterization of a class of going-down rings with zero-divisors [14, Corollary 2.6], there has been considerable interest in the class of rings  $R$  such that  $Z(R) = Nil(R)$  as a generalization of the class of domains. In part, this has been possible technically because the total quotient ring of any such  $R$  is conveniently described as a localization of  $R$  [14, Proposition 2.3] and the localization of such  $R$  at any of its prime ideals can be viewed as an overring of  $R$  [10, Proposition 2.5 (a)]. The next result contributes to the theme of extending results on domains to the context of rings  $R$  satisfying  $Z(R) = Nil(R)$  by generalizing parts (i) and (ii) of [15, Theorem 2.3]. As parts (c) and (d) illustrate, our best analogues of domain-theoretic results arise in case  $R \in \mathcal{H}$ .

Note that if a ring  $R$  satisfies  $Z(R) = Nil(R)$ , then  $R \in \mathcal{H}$  if and only if  $Nil(R)$  (which is necessarily a prime ideal of  $R$ ) is divided. It is trivial that if  $R \in \mathcal{H}$  and  $Z(R) = Nil(R)$ , then  $\phi_R : Tot(R) \rightarrow R_{Nil(R)}$  is the identity map, and so parts of Theorem 2.5 could be formulated  $\phi$ -theoretically.

**Theorem 2.5.** *Let  $R$  be a ring such that  $Z(R) = Nil(R)$ , and let  $P$  be a prime ideal of  $R$ . Put  $D := R_{red}$  and  $\bar{P} := P/Nil(R)$ . Then:*

(a) *If  $R \subseteq R_P$  is a strong extension, then the set of prime ideals of  $R$  which contain  $P$  is linearly ordered (by inclusion) and  $R$  is quasilocal.*

(b) If  $P$  is a divided prime ideal of  $R$ , then  $\bar{P}$  is a divided prime ideal of  $D$ .

(c) Suppose also that either  $R \subseteq R_P$  is a strong extension or  $R \in \mathcal{H}$ . Then  $\bar{P}$  is a divided prime ideal of  $D$  if and only if  $P$  is a divided prime ideal of  $R$ .

(d) Consider the following two conditions:

(1)  $P$  is a divided prime ideal of  $R$  and  $R/P$  is a PVD;

(2)  $R \subseteq R_P$  is a strong extension.

Then (2)  $\Rightarrow$  (1); and if  $R \in \mathcal{H}$ , then (1)  $\Leftrightarrow$  (2).

PROOF. Note that  $Z(R) = Nil(R)$  is the unique minimal prime ideal of  $R$  (cf. [14, Proposition 2.3 (b)]). In particular,  $D$  is a domain. Moreover, since localization commutes with the formation of factor rings, we have a canonical identification  $D_{\bar{P}} = R_P/(Nil(R)R_P)$ .

(a) Suppose that  $R \subseteq R_P$  is a strong extension. Then, by Theorem 2.1,  $Nil(R) = Nil(R_P)$  and  $D \subseteq E := R_P/Nil(R)$  is a strong extension of rings. The first of these facts ensures that  $Nil(R)$  is an ideal of  $R_P$ , whence  $Nil(R) = Nil(R)R_P$ . Therefore, the above comments allow us to identify  $E$  with  $D_{\bar{P}}$ , which is an overring of the domain  $D$  and, hence, is itself a domain. Applying the domain-theoretic result [15, Theorem 2.3 (ii)] to the extension  $D \subseteq E$  of domains, we conclude that the set of prime ideals of  $D$  which contain  $\bar{P}$  is linearly ordered and  $D$  is quasilocal. The assertion then follows easily from standard homomorphism theorems and the fact that each prime ideal contains a minimal prime ideal [18, Theorem 10].

(b) Our task is to prove that if  $\bar{x} \in D \setminus \bar{P}$ , then  $\bar{P} \subseteq D\bar{x}$ . We can write  $\bar{x} = x + Nil(R)$ , with  $x \in R \setminus P$ . As  $P$  is assumed to be a divided prime ideal of  $R$ , we have that  $P \subseteq Rx$ . Therefore,

$$\bar{P} = P/Nil(R) \subseteq (Rx + Nil(R))/Nil(R) = (R_{red})(x + Nil(R)) = D\bar{x}.$$

(c) Since  $Z(R) = Nil(R)$ , [10, Proposition 2.5 (a)] allows  $R_P$  to be viewed (up to  $R$ -algebra isomorphism) as an overring of  $R$ . Hence, by [10, Lemma 2.6],  $Z(R_P) = Nil(R_P)$ . Accordingly, by [10, Proposition 2.5 (c)],  $P$  (resp.,  $\bar{P}$ ) is a divided prime ideal of  $R$  (resp.,  $D$ ) if and only if  $PR_P = P$  (resp.,  $\bar{P}D_{\bar{P}} = \bar{P}$ ). Thus, by the above identification of  $D_{\bar{P}}$ , it follows that  $\bar{P}$  is a divided prime ideal of  $D$  if and only if  $PR_P/(Nil(R)R_P)$  is canonically identified with  $P/Nil(R)$ .

We claim that, under the hypotheses of (c),  $Nil(R)R_P = Nil(R)$ . Indeed, if  $R \subseteq R_P$  is a strong extension, this was established in the proof of (a). If, on the other hand,  $R \in \mathcal{H}$ , then  $Nil(R) \subseteq Rx$  for each  $x \in R \setminus Nil(R)$ , and so working in the overring  $R_P$  of  $R$ , we have  $Nil(R)R_P = \cup\{Nil(R)x^{-1} \mid x \in R \setminus P\} \subseteq R \cap Nil(R_P) = Nil(R)$ . As the reverse inclusion is trivial, we have established the claim in all cases. Therefore,  $\bar{P}$  is a divided prime ideal of  $D$  if and only

if  $PR_P/Nil(R)$  is canonically identified with  $P/Nil(R)$ ; that is, by a standard homomorphism theorem, if and only if  $PR_P = P$ ; that is, if and only if  $P$  is a divided prime ideal of  $R$ .

(d) (2)  $\Rightarrow$  (1): Assume (2). By Theorem 2.1,  $Nil(R) = Nil(R_P)$  and  $D \subseteq R_P/Nil(R)$  is a strong extension. As  $Nil(R) = Nil(R)R_P$  in this case, the above description of  $D_{\overline{P}}$  therefore shows that  $D \subseteq D_{\overline{P}}$  is a strong extension. This means, according to a domain-theoretic result [15, Theorem 2.3 (i)], that  $\overline{P}$  is a divided prime ideal of  $D$  and  $D/\overline{P}$  is a PVD. By (c),  $P$  is a divided prime ideal of  $R$ . Since  $D/\overline{P} \cong R/P$ , (1) follows.

Assume (1). By (b),  $\overline{P}$  is a divided prime ideal of  $D$ . As  $D/\overline{P} \cong R/P$ , [15, Theorem 2.3 (i)] implies that  $D \subseteq D_{\overline{P}}$  is a strong extension. In other words,  $R_{red} \subseteq R_P/(Nil(R)R_P)$  is a strong extension. Thus, by Theorem 2.1, (2) will follow if we show that  $R \in \mathcal{H}$  implies  $Nil(R) = Nil(R)R_P = Nil(R_P)$ . The first of these equations was, in fact, already shown in the proof of (c); and the second of these equations is an easy consequence of the fact that  $Z(R) = Nil(R) \subseteq P$ . The proof is complete.  $\square$

It is interesting to note that the proof of Theorem 2.5 (b) did not use the hypothesis that  $Z(R) = Nil(R)$ . The next example shows that the converse of Theorem 2.5 (b) fails, and so the hypothesis of Theorem 2.5 (c) cannot be significantly weakened.

**Example 2.6.** *There exist a ring  $R$  and a prime ideal  $P$  of  $R$  such that  $Z(R) = Nil(R)$  and  $\overline{P} := P/Nil(R)$  is a divided prime ideal of  $R_{red}$ , but  $P$  is not a divided prime ideal of  $R$ . It can be further arranged that  $R_{red}$  is (isomorphic to) any given domain which is not a field.*

PROOF. We specialize the second construction in Example 2.3. Let  $D$  be any domain which is not a field, and let  $R := D[X]/(X^2) = D \oplus Dx$ , where  $x := X + (X^2)$  satisfies  $x^2 = 0$ . Since  $D$  is a domain, it is easy to check that  $Z(R) = Dx = Nil(R)$ , and so we have a canonical isomorphism  $R_{red} = (D \oplus Dx)/Dx \cong D$ . Put  $P := Nil(R)$ . Then  $P$  is a prime ideal of  $R$ , and of course,  $\overline{P} = P/Nil(R) = 0$  is a divided prime ideal of (the domain)  $R_{red}$ . However,  $P$  is not divided in  $R$ . Indeed, if we pick a nonzero element  $a \in D \setminus U(D)$  and any  $b \in D$ , then  $c := a + bx \in R \setminus P$  but  $P \not\subseteq Dc$ . To see this, suppose, on the contrary, that  $x = (d_1 + d_2x)c$ , for some  $d_1, d_2 \in D$ . Then  $d_1a = 0$  and  $d_1b + d_2a = 1$ , whence  $d_1 = 0$  and  $d_2a = 1$ , contradicting the choice of  $a$  as a nonunit.  $\square$

In comparing Theorem 2.5 with Example 2.6, one may hope that rings in  $\mathcal{H}$  would have their divided prime ideals exhibiting better behavior. The next result

shows that this hope is realized and demonstrates another way in which  $\mathcal{H}$  is a more tractable working hypothesis than “ $Z(R) = Nil(R)$ ”.

**Proposition 2.7.** *Let  $R \in \mathcal{H}$ , with  $\phi = \phi_R$  denoting the canonical ring homomorphism  $Tot(R) \rightarrow R_{Nil(R)}$ . Let  $P$  be any prime ideal of  $R$ . Then  $P$  is a divided prime ideal of  $R$  if and only if  $\phi(P)$  is a divided prime ideal of  $\phi(R)$ .*

PROOF. We omit the easy proof of the “only if” assertion. Conversely, suppose that  $\phi(P)$  is divided in  $\phi(R)$ . We will show that  $P$  is divided in  $R$  by proving that if  $r \in R \setminus P$  and  $p \in P$ , then  $p \in Rr$ . We claim that  $\phi(r) \in \phi(R) \setminus \phi(P)$ . Otherwise,  $\phi(r) = \phi(q)$  for some  $q \in P$ , so that  $r - q \in \ker(\phi) \subseteq Nil(R) \subseteq P$  and  $r = (r - q) + q \in P$ , a contradiction. This proves the above claim. Therefore, since  $\phi(P)$  is divided in  $\phi(R)$ , we have  $\phi(P) \subseteq \phi(R)\phi(r)$ . In particular,  $\phi(p) = \phi(s)\phi(r) = \phi(sr)$  for some  $s \in R$ . Then  $w := p - sr \in \ker(\phi) \subseteq Nil(R)$ . As  $r \notin Nil(R)$  (since  $r \notin P$ ) and  $R \in \mathcal{H}$ , we have  $Nil(R) \subseteq Rr$  and, in particular,  $w = tr$  for some  $t \in R$ . Hence,  $p = w + sr = (t + s)r \in Rr$ , as required.  $\square$

Recall that for any ring  $R \in \mathcal{H}$ ,  $\phi_R$  denotes the canonical ring homomorphism  $Tot(R) \rightarrow R_{Nil(R)}$ . The next result collects some useful facts about the rings in  $\mathcal{H}$ .

**Lemma 2.8.** *Let  $R \in \mathcal{H}$  and let  $P$  be any prime ideal of  $R$ . Then:*

- (a)  $R_P \in \mathcal{H}$ .
- (b)  $R/P$  is ring-isomorphic to  $\phi(R)/\phi(P)$ .
- (c)  $\phi|_R$  induces ring-isomorphisms of domains  $R_{red} \rightarrow \phi_R(R)_{red}$  and  $(R_P)_{red} \rightarrow \phi_{R_P}(R_P)_{red}$ .
- (d)  $\phi_{R_P}(R_P) = \phi_R(R)_{\phi_R(P)}$  is an overring of  $\phi_R(R)$ .
- (e)  $Nil(\phi_{R_P}(R_P)) = Nil(\phi_R(R))$ .

PROOF. (a) By hypothesis,  $Nil(R)$  is a divided prime ideal of  $R$ . Hence,  $Nil(R_P) = Nil(R)R_P$  is a prime ideal of  $R_P$ . To show that it is also divided, we consider  $\xi \in R_P \setminus Nil(R_P)$  and need to show that  $Nil(R)R_P \subseteq R_P\xi$ . We can write  $\xi = \frac{r}{z}$  for some  $r \in Nil(R)$  and  $z \in R \setminus P$ . Since  $Nil(R)$  is divided in  $R$ , we have  $Nil(R) \subseteq Rr$ , whence  $Nil(R)R_P \subseteq RrR_P = \frac{r}{z}zR_P = \xi R_P$ , as desired.

(b) This is [1, Lemma 2.5].

(c) It suffices to establish the first isomorphism, as one may then, by (a), repeat the argument with  $R$  replaced by  $R_P$ . Note that

$$\phi_R(R)_{red} := \phi_R(R)/Nil(\phi_R(R)) = \phi_R(R)/\phi_R(Nil(R))$$

which, by (b), is canonically identified with  $R/Nil(R) =: R_{red}$ .

(d) Note that the assertion is meaningful by virtue of (a). Now, [2, Lemma 3.8]



handles the case where  $P$  is a nonnil prime ideal of  $R$ . Moreover, if  $P = \text{Nil}(R)$ , then  $\phi_{R_P}(R_P) = R_P = \text{Tot}(\phi_R(R))$  is also an overring of  $\phi_R(R)$ .

(e) By (d),

$$\begin{aligned} \text{Nil}(\phi_{R_P}(R_P)) &= \text{Nil}(\phi_R(R)_{\phi_R(P)}) = \text{Nil}(\phi_R(R))\phi_R(R)_{\phi_R(P)} = \\ &= \phi_R(\text{Nil}(R))\phi_R(R)_{\phi_R(P)} = \phi_R(\text{Nil}(R))_{\phi_R(P)}. \end{aligned}$$

By the proof of (d) (see the proof of [2, Lemma 3.8]), the last-displayed expression is the same as  $\phi_{R_P}(\text{Nil}(R)_P)$ . We claim that this, in turn, can be simplified as  $\phi_R(\text{Nil}(R))$ .

It is clear that  $\phi_R(\text{Nil}(R)) \subseteq \phi_{R_P}(\text{Nil}(R)_P)$ . For the reverse inclusion, we will show that if  $u \in \text{Nil}(R)$  and  $v \in R \setminus P$ , then there exists  $w \in \text{Nil}(R)$  such that  $\frac{u}{v} = \frac{w}{1} \in R_P$ . Indeed, since  $\text{Nil}(R)$  is divided in  $R$ , we have  $\text{Nil}(R) \subseteq Rv$ , and so there exists  $w \in R$  such that  $u = vw$ . As  $\text{Nil}(R)$  is a prime ideal of  $R$ , it follows that  $w \in \text{Nil}(R)$ . Since  $\frac{u}{v} = \frac{w}{1}$ , this completes the proof of the above claim. Then the proof is complete since  $\phi_R(\text{Nil}(R)) = \text{Nil}(\phi_R(R))$ .  $\square$

One consequence of Proposition 2.7 is that if  $R \in \mathcal{H}$ , then  $R$  is a divided ring (in the sense of [8]) if and only if  $\phi_R(R)$  is a divided ring. When coupled with Lemma 2.8 (d), this implies that if  $R \in \mathcal{H}$ , then  $R$  is a locally divided ring (in the sense of [10]) if and only if  $\phi_R(R)$  is a locally divided ring.

**Theorem 2.9.** *Let  $R \in \mathcal{H}$  and let  $P$  be a prime ideal of  $R$ . Then:*

(a) *Consider the domain  $D := R_{\text{red}}$  and the prime ideal  $\bar{P} := P/\text{Nil}(R)$  of  $D$ .*

*Then the following conditions are equivalent:*

- (1)  *$P$  is a divided prime ideal of  $R$  and  $R/P$  is a PVD;*
- (2)  *$D \subseteq D_{\bar{P}}$  is a strong extension;*
- (3)  *$\phi_R(R) \subseteq \phi_{R_P}(R_P)$  is a strong extension.*

(b) *If  $\phi_R(R) \subseteq \phi_{R_P}(R_P)$  is a strong extension, then the set of prime ideals of  $R$  which contain  $P$  is linearly ordered (by inclusion) and  $R$  is quasilocal.*

PROOF. (a) By [15, Theorem 2.3 (i)], (2)  $\Leftrightarrow \bar{P}$  is a divided prime ideal of  $D$  and  $D/\bar{P}$  is a PVD. In view of Theorem 2.5 (c) and the standard isomorphism  $D/\bar{P} \cong R/P$ , we thus have that (2)  $\Leftrightarrow$  (1).

Now, by Theorem 2.1 and Lemma 2.8 (e), we have that (3)  $\Leftrightarrow \phi_R(R)_{\text{red}} \subseteq \phi_{R_P}(R_P)_{\text{red}}$  is a strong extension. By Lemma 2.8 (c), this last condition is equivalent to  $D \subseteq (R_P)_{\text{red}}$  being a strong extension. Thus, to prove that (3)  $\Leftrightarrow$  (2), it suffices to produce a canonical isomorphism  $(R_P)_{\text{red}} \cong D_{\bar{P}}$ . For this, observe that

$$D_{\bar{P}} = (R/\text{Nil}(R))_{P/\text{Nil}(R)} \cong R_P/(\text{Nil}(R)R_P) = R_P/\text{Nil}(R_P) =: (R_P)_{\text{red}}.$$

(b) By (a),  $P$  is a divided prime ideal of  $R$  and  $R/P$  is a PVD. The first of these conditions ensures that  $P$  is contained in each maximal ideal of  $R$ ; and the second ensures (by [16, Corollary 1.3]) that  $R/P$  is quasilocal and that the set of prime ideals of  $R/P$  is linearly ordered by inclusion. It follows by a standard homomorphism theorem that  $R$  is quasilocal and that the set of prime ideals of  $R$  which contain  $P$  is linearly ordered by inclusion.  $\square$

Note that in formulating Theorem 2.9 (a) for the context of rings in  $\mathcal{H}$ , rather than for rings in which each zero-divisor is nilpotent, we did not consider the condition “ $R \subseteq R_P$  is a strong extension” for the simple reason that the canonical ring homomorphism  $R \rightarrow R_P$  need not be an injection. To recover a context where something like  $R \rightarrow R_P$  is an inclusion (and, hence, possibly a strong extension), we had recourse to a  $\phi$ -theoretic formulation, in view of the “overring” conclusion in Lemma 2.8 (d).

We next infer one of our main applications, an analogue of [15, Theorem 2.9].

**Corollary 2.10.** *Let  $R \in \mathcal{H}$ . Then  $R$  is a  $\phi$ -PVR if and only if  $R$  has a prime ideal  $P$  satisfying the following two conditions:*

- (i)  $\phi_R(R) \subseteq \phi_{R_P}(R_P)$  is a strong extension;
- (ii)  $R_P$  is a  $\phi$ -chained ring.

PROOF. By [6, Proposition 2.9],  $R$  is a  $\phi$ -PVR if and only if (the domain)  $D := R_{red}$  is a PVD. Now, according to [15, Theorem 2.9],  $D$  is a PVD if and only if there exists a prime ideal  $P$  of  $R$  such that the prime ideal  $\bar{P} := P/Nil(R)$  of  $R/Nil(R) =: R_{red}$  satisfies the following two conditions:

- (a)  $D \subseteq D_{\bar{P}}$  is a strong extension;
- (b)  $D_{\bar{P}}$  is a valuation domain.

As noted in the Introduction, it was shown in [1, Theorem 2.7] that (ii)  $\Leftrightarrow (R_P)_{red}$  is a valuation domain. Thus, (ii)  $\Leftrightarrow$  (b), since it was shown in the proof of Theorem 2.9 (a) that  $(R_P)_{red} \cong D_{\bar{P}}$ . As Theorem 2.9 (a) also established that (i)  $\Leftrightarrow$  (a), the proof is complete.  $\square$

The proof of Corollary 2.10 also establishes the following result.

**Corollary 2.11.** *Let  $R \in \mathcal{H}$  and set  $D := R_{red}$ . Then  $R$  is a  $\phi$ -PVR if and only if  $R$  has a prime ideal  $P$  satisfying the following two conditions:*

- (i)  $D \subseteq D_{P/Nil(R)}$  is a strong extension;
- (ii)  $D_{P/Nil(R)}$  is a valuation domain.

**Corollary 2.12.** *Let  $R \in \mathcal{H}$  such that  $Z(R) = Nil(R)$ . Then  $R$  is a  $\phi$ -PVR if and only if  $R$  has a prime ideal  $P$  satisfying the following two conditions:*

- (i)  $R \subseteq R_P$  is a strong extension;
- (ii)  $R_P$  is a  $\phi$ -chained ring.

PROOF.  $\phi_R(R) = R$ , since the hypotheses ensure that  $\phi_R$  is the identity map. Then  $\phi_{R_P}(R_P) = R_P$  by Lemma 2.8 (d). An application of Corollary 2.10 completes the proof.  $\square$

Recall that a domain  $R$  is said to be *Archimedean* in case  $\bigcap_{n=1}^{\infty} Rr^n = 0$  for each  $r \in R \setminus U(R)$ . The most natural examples of Archimedean domains are arbitrary Noetherian domains and the domains of (Krull) dimension at most 1. By analogy with the preceding definition, we say that a ring  $R$  is a *nonnil-Archimedean ring* if  $\bigcap_{n=1}^{\infty} Rr^n = \text{Nil}(R)$  for each  $r \in R \setminus (\text{Nil}(R) \cup U(R))$ . Thus, any Noetherian ring is nonnil-Archimedean, as is any ring of dimension at most 1. We leave the proof of the following easy result to the reader.

**Lemma 2.13.** *Let  $R \in \mathcal{H}$ . Then  $R$  is a nonnil-Archimedean ring if and only if  $R_{red}$  is an Archimedean domain.*

We next provide some additional examples of nonnil-Archimedean rings by giving the following analogue of [15, Proposition 2.11].

**Proposition 2.14.** *Let  $R \in \mathcal{H}$  be a ring which is not zero-dimensional, and put  $D := R_{red}$ . Then the following conditions are equivalent:*

- (1)  $R$  is a nonnil-Archimedean ring and, for some prime ideal  $P$  of  $R$ , the proper extension  $\phi_R(R) \subset \phi_{R_P}(R_P)$  is a strong extension;
- (2)  $R$  is a one-dimensional  $\phi$ -PVR;
- (3)  $D$  is a one-dimensional PVD;
- (4)  $D$  is an Archimedean domain and, for some prime ideal  $P$  of  $R$ , the proper extension  $D \subset D_{P/\text{Nil}(R)}$  is a strong extension.

PROOF. Note that  $\dim(R) = \dim(R_{red}) = \dim(D)$  on general principles. In particular, the domain  $D$  is not a field.

(2)  $\Leftrightarrow$  (3): In view of the above comments, it suffices to recall from [6, Proposition 2.9] that  $R$  is a  $\phi$ -PVR if and only if  $D$  is a PVD.

(3)  $\Leftrightarrow$  (4): Apply [15, Proposition 2.11].

(4)  $\Leftrightarrow$  (1): By Lemma 2.13, Theorem 2.9 (a) and Lemma 2.8 (c) (as well as the proofs of these last two results), (4) is equivalent to the following condition:  $R$  is a nonnil-Archimedean ring and, for some prime ideal  $P$  of  $R$ , the proper extension  $\phi_R(R)_{red} \subset \phi_{R_P}(R_P)_{red}$  is a strong extension. In view of Lemma 2.8 (e) and Theorem 2.1, this last condition is equivalent to (1), and so the proof is complete.  $\square$

By reasoning as in the proof of Corollary 2.12, one immediately infers the following special case of Proposition 2.14.

**Corollary 2.15.** *Let  $R \in \mathcal{H}$  be a ring which is not zero-dimensional and which satisfies  $Z(R) = \text{Nil}(R)$ . Put  $D := R_{\text{red}}$ . Then the following conditions are equivalent:*

- (1)  *$R$  is a nonnil-Archimedean ring and, for some prime ideal  $P$  of  $R$ , the proper extension  $R \subset R_P$  is strong;*
- (2)  *$R$  is a one-dimensional  $\phi$ -PVR;*
- (3)  *$D$  is a one-dimensional PVD.*

Recall from [2] that a ring  $R \in \mathcal{H}$  is called  $\phi$ -completely integrally closed if  $\phi(R)$  is completely integrally closed in  $\text{Tot}(\phi(R)) (= R_{\text{Nil}(R)})$ . It is known that a ring  $R \in \mathcal{H}$  is  $\phi$ -completely integrally closed if and only if the domain  $R_{\text{red}}$  is a completely integrally closed domain [2, Lemma 2.8]. The following result is an analogue of [15, Proposition 2.11(bis)]. Its proof can be modelled after that of Proposition 2.14, bearing in mind the above comments, the fact that a ring  $R \in \mathcal{H}$  is a  $\phi$ -chained ring if and only if  $R_{\text{red}}$  is a valuation domain [1, Theorem 2.7], and [15, Proposition 2.11(bis)] itself.

**Proposition 2.16.** *Let  $R \in \mathcal{H}$  be a ring which is not zero-dimensional, and put  $D := R_{\text{red}}$ . Then the following conditions are equivalent:*

- (1)  *$R$  is a  $\phi$ -completely integrally closed ring and, for some prime ideal  $P$  of  $R$ , the proper extension  $\phi_R(R) \subset \phi_{R_P}(R_P)$  is a strong extension;*
- (2)  *$R$  is a one-dimensional  $\phi$ -chained ring;*
- (3)  *$D$  is a one-dimensional valuation domain;*
- (4)  *$D$  is a completely integrally closed domain and, for some prime ideal  $P$  of  $R$ , the proper extension  $D \subset D_{P/\text{Nil}(R)}$  is a strong extension.*

By reasoning as in the proof of Corollary 2.12, one immediately infers the following special case of Proposition 2.16.

**Corollary 2.17.** *Let  $R \in \mathcal{H}$  be a ring which is not zero-dimensional and which satisfies  $Z(R) = \text{Nil}(R)$ . Put  $D := R_{\text{red}}$ . Then the following conditions are equivalent:*

- (1)  *$R$  is a completely integrally closed ring and, for some prime ideal  $P$  of  $R$ , the proper extension  $R \subset R_P$  is a strong extension;*
- (2)  *$R$  is a one-dimensional  $\phi$ -chained ring;*
- (3)  *$D$  is a one-dimensional valuation domain.*

Next, we give an analogue of [15, Proposition 2.14].

**Proposition 2.18.** *Let  $R \subseteq T$  be an integral extension of rings. Then  $R \subseteq T$  is a strong extension if and only if  $\text{Spec}(R) = \text{Spec}(T)$  (as sets).*

PROOF. The ‘if’ assertion is trivial (and does not require the hypothesis of integrality). The ‘only if’ assertion can be established as in the proof of [13, Proposition 4.1].  $\square$

We close this section with the following analogue of [15, Proposition 2.7].

**Proposition 2.19.** *Let  $(B, M)$  be a quasilocal ring such that  $\text{Nil}(B)$  is a prime ideal of  $B$ , and let  $A$  be any domain having  $K := B/M$  as its quotient field. Let  $R$  be the pullback of the diagram*

$$\begin{array}{ccc} R & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & B/M \end{array}$$

in which the horizontal arrows are the usual inclusion maps and are denoted by  $u$ , and the vertical arrows are the usual surjective maps and are denoted by  $v$ . Let  $S$  be any ring such that  $R \subseteq S \subseteq B$ . Then:

- (a)  $S \subseteq B$  is a strong extension if and only if  $v(S)$  is a PVD.
- (b)  $R \subseteq S$  is a strong extension if and only if  $A \subseteq v(S)$  is a strong extension.

PROOF. If  $b \in \text{Nil}(B)$ , then  $b^n = 0$  for some positive integer  $n$ , and so  $v(b)^n = v(b^n) = v(0) = 0$ , whence  $v(b) = 0 \in A$  and, since  $R$  is a pullback,  $b \in v^{-1}(A) = R$ . It follows that  $\text{Nil}(B) = \text{Nil}(R)$ . Then  $\text{Nil}(S) = \text{Nil}(B)$  also.

Now, let  $B_1 := B_{red}$ ,  $M_1 := M/\text{Nil}(B)$ ,  $K_1 := B_1/M_1$ ,  $A_1$  the canonical image of  $A$  inside  $K_1$ , and  $R_1$  the pullback of the following diagram

$$\begin{array}{ccc} R_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1/M_1, \end{array}$$

in which the horizontal arrows are the usual inclusion maps and are denoted by  $u_1$ , and the vertical arrows are the usual surjective maps and are denoted by  $v_1$ . Observe that  $K_1 \cong K$ ,  $A_1 \cong A$ , and  $K_1$  is the quotient field of  $A_1$ . Note that  $R_1 \subseteq S_1 := S/\text{Nil}(R) \subseteq B_1$ . Moreover,  $B_1$  is a domain since  $\text{Nil}(B)$  is a prime ideal of  $B$ .

According to [15, Proposition 2.7] (i),  $S_1 \subseteq B_1$  is a strong extension if and only if  $v_1(S_1)$  is a PVD. Therefore, in order to prove (a), it suffices to observe the following two points:  $S_1 (= S_{red}) \subseteq B_1$  is a strong extension if and only if  $S \subseteq B$  is a strong extension (by Theorem 2.1); and one has canonical isomorphisms

$$v_1(S_1) = S_1/(M/\text{Nil}(B)) \cong S/M = v(S).$$

To prove (b), recall first the corresponding domain-theoretic fact [15, Proposition 2.7] (ii). This result ensures that  $R_1 \subseteq S_1$  is a strong extension if and only if  $A_1 \subseteq v_1(S_1)$  is a strong extension. In view of the above comments, we may use Theorem 2.1 and argue as above, provided that we find a canonical isomorphism  $R_1 \cong R_{red}$ . To that end, note that viewing  $A_1$  as the image of the composite

$$A \hookrightarrow K = B/M \cong (B/Nil(B))/(M/Nil(B)) = B_1/M_1 = K_1,$$

we see that  $A_1 = \{(b + Nil(B)) + M_1 \in B_1/M_1 \mid b + M \in A\}$ . Therefore,

$$\begin{aligned} R_1 &= \{b + Nil(B) \in B_1 \mid v_1(b + Nil(B)) \in A_1\} = \\ &\{b + Nil(B) \in B_1 \mid (b + Nil(B)) + M_1 \in A_1\} = \{b + Nil(B) \in B_1 \mid b + M \in A\} = \\ &\{b + Nil(B) \in B_1 \mid v(b) \in A\} = \{b + Nil(B) \mid b \in R\} = R_{red}, \end{aligned}$$

thus completing the proof.  $\square$

### 3. THE NONOVERRING CASE

This section is devoted to analyzing strong extensions that are not necessarily overrings. Its main result is a  $\phi$ -theoretic analogue of [15, Theorem 3.1]. First, we give two lemmas which address concerns that were trivial in the domain-theoretic context of [15]. Lemma 3.1 will be used in the proof of Theorem 3.3 and Lemma 3.2 characterizes one of the hypotheses appearing in Theorem 3.3 (a), (d).

**Lemma 3.1.** *Let  $A$  be a subring of a ring  $C$ . Then:*

(a) *Let  $B := A_{A \setminus Z(A)}$ , the total quotient ring of  $A$ . Then there exists a ring  $D$  such that  $B$  and  $C$  are each (unital) subrings of  $D$  if and only if each non-zero-divisor of  $A$  is a non-zero-divisor in  $C$ .*

(b) *If  $A \subseteq C$  is a strong extension, then each non-zero-divisor of  $A$  is a non-zero-divisor in  $C$ .*

(c) *If  $A \subseteq C$  is a strong extension, then there exists a ring  $D$  such that the total quotient ring of  $A$  and  $C$  are each (unital) subrings of  $D$ .*

**PROOF.** (a) Suppose first that there exists a ring  $D$  that has both  $B$  and  $C$  as (unital) subrings. We will show that if  $a \in A \setminus Z(A)$  and  $ac = 0$  for some  $c \in C$ , then  $c = 0$ . Since  $a$  has a multiplicative inverse  $a^{-1} = \frac{1}{a}$  in  $B$ , we have that  $c = (ac)a^{-1} = 0a^{-1} = 0$  (in  $D$  and, hence, in  $C$ ).

Conversely, suppose that each non-zero-divisor of  $A$  is a non-zero-divisor in  $C$ . Consider the ring of fractions  $D := C_{A \setminus Z(A)}$ . The assignment  $c \mapsto \frac{c}{1}$  defines a (unital) ring homomorphism  $f : C \rightarrow D$ . It is easy to see that the hypothesis (on non-zero-divisors of  $A$ ) ensures that  $f$  is an injection. For convenience, we use  $f$  to identify  $C$  as a subring of  $D$ . Next, by applying the universal mapping property of

the ring of fractions  $A_{A \setminus Z(A)} = B$ , obtain an (unital)  $R$ -algebra homomorphism  $g : B \rightarrow D$ . Specifically,  $g(\frac{a_1}{a_2}) = \frac{a_1}{a_2}$  for all  $a_1 \in A$  and all  $a_2 \in A \setminus Z(A)$ . Once again, it is easy to see that the hypothesis ensures that  $g$  is an injection. Next, it is straightforward to use the hypothesis to verify that we can use  $g$  to identify  $B$  as a subring of  $D$  without altering the identification that was effected earlier by  $f$ . This completes the proof of (a).

(b) We will show that if  $a \in A \setminus Z(A)$  and  $ac = 0$  for some  $c \in C$ , then  $c = 0$ . Since  $a$  is not nilpotent, there exists a prime ideal  $P$  of  $A$  such that  $a \notin P$  (cf. [18, p.16]). As  $ac = 0 \in P$  and  $P$  is  $C$ -strong with  $a \notin P$ , it follows that  $c \in P$ . In particular,  $c \in A$ , whence  $c = 0$  since  $a$  is not a zero-divisor of  $A$ .

(c) Combine (b) and (a). □

**Lemma 3.2.** *Let  $R$  be a ring. Consider the following four conditions:*

- (1) *No prime ideal of  $\phi(R)$  contains a non-zero-divisor of  $\phi(R)$ ;*
- (2)  *$\phi(R)$  has a unique prime ideal;*
- (3)  *$R$  has a unique prime ideal;*
- (4) *No prime ideal of  $R$  contains a non-zero-divisor of  $R$ .*

*Then:*

- (a) *If  $R \in \mathcal{H}$ , then (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3).*
- (b) *(3)  $\Rightarrow$  (4).*
- (c) *If  $Z(R) = Nil(R)$ , then (4)  $\Rightarrow$  (3).*
- (d) *If  $R \in \mathcal{H}$  and  $Z(R) = Nil(R)$ , then (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4).*

PROOF. (a) Assume that  $R \in \mathcal{H}$ . Then, since  $\ker(\phi) \subseteq Nil(R)$ , the assignment  $P \mapsto \phi(P)$  determines an order-isomorphism between the set of prime ideals of  $R$  and the set of prime ideals of  $\phi(R)$ . Thus, (3)  $\Leftrightarrow$  (2). However, since  $\phi(R) \in \mathcal{H}$ , the unique minimal prime ideal of  $\phi(R)$  is  $Nil(\phi(R)) = Z(\phi(R))$ . It follows that (2)  $\Rightarrow$  (1); and, since each prime ideal of  $\phi(R)$  contains  $Nil(\phi(R))$ , that (1)  $\Rightarrow$  (2).

(b) Suppose that  $R$  has a unique prime ideal  $P$ . Then  $P$  is necessarily a minimal prime ideal of  $R$ , whence  $P \subseteq Z(R)$  by [18, Theorem 84]. It is then clear that (4) holds.

(c), (d) It remains only to prove that if  $Z(R) = Nil(R)$ , then (4)  $\Rightarrow$  (3). To do so, note that  $Z(R)$  is then the unique minimal prime ideal of  $R$ , and replace “ $\phi(R)$ ” with “ $R$ ” in the proof in (a) that (1)  $\Rightarrow$  (2). The proof is complete. □

As in [18, p. 28], it will be convenient to let LO denote the lying-over property of ring extensions.

**Theorem 3.3.** *Let  $R \in \mathcal{H}$  and let  $T$  be any ring that contains  $\phi(R)$  as a (unital) subring. Then:*

(a) *Assume also that  $(\text{Tot}(\phi(R)) =)R_{\text{Nil}(R)} \subseteq T$ . Then:*

(a1) *If, in addition,  $R$  has more than one prime ideal and  $\phi(R) \subseteq T$  is a strong extension, then  $R$  is a  $\phi$ -PVR and  $U(T) = U(R_{\text{Nil}(R)})$ .*

(a2) *If, in addition,  $T$  has only one minimal prime ideal and both  $R$  is a  $\phi$ -PVR and  $U(T) = U(R_{\text{Nil}(R)})$ , then  $\phi(R) \subseteq T$  is a strong extension.*

(a3) *If, in addition,  $R$  has more than one prime ideal and  $T$  has only one minimal prime ideal, then  $\phi(R) \subseteq T$  is a strong extension if and only if both  $R$  is a  $\phi$ -PVR and  $U(T) = U(R_{\text{Nil}(R)})$ .*

(b) *If  $\phi(R) \subseteq T$  is a strong extension, then both  $\phi(R) \subseteq T \cap R_{\text{Nil}(R)}$  and  $T \cap R_{\text{Nil}(R)} \subseteq T$  are strong extensions.*

(c) *Assume also that each non-zero-divisor of  $\phi(R)$  is a non-zero-divisor in  $T$ . If  $\phi(R) \subseteq T \cap R_{\text{Nil}(R)}$  satisfies LO and both  $\phi(R) \subseteq T \cap R_{\text{Nil}(R)}$  and  $T \cap R_{\text{Nil}(R)} \subseteq T$  are strong extensions, then  $\phi(R) \subseteq T$  is a strong extension.*

(d) *Assume also that  $R$  has more than one prime ideal and that each non-zero-divisor of  $\phi(R)$  is a non-zero-divisor in  $T$ . If  $\phi(R) \subseteq T$  is a strong extension such that  $T \cap R_{\text{Nil}(R)} = \phi(R)$ , then  $U(T) = U(\phi(R))$ .*

PROOF. (a) Consider the domain  $D := R_{\text{red}}$ , canonically identified with  $\phi(R)_{\text{red}}$  as in Lemma 2.8 (c), put  $K := R_{\text{Nil}(R)}$ , and recall that  $K_{\text{red}}$  can be viewed as the quotient field of  $D$ . Observe that we have the tower of rings  $D \subseteq K_{\text{red}} \subseteq T_{\text{red}}$ . We claim that it follows from [15, Theorem 3.1 (i)] that we have the following three facts:

( $\alpha$ ) If, in addition,  $R$  has more than one prime ideal and  $D \subseteq T_{\text{red}}$  is a strong extension, then  $D$  is a PVD and  $U(T_{\text{red}}) = U(K_{\text{red}})$ .

( $\beta$ ) If, in addition,  $T$  has only one minimal prime ideal and both  $D$  is a PVD and  $U(T_{\text{red}}) = U(K_{\text{red}})$ , then  $D \subseteq T_{\text{red}}$  is a strong extension.

( $\gamma$ ) If, in addition,  $R$  has more than one prime ideal and  $T$  has only one minimal prime ideal, then  $D \subseteq T_{\text{red}}$  is a strong extension if and only if both  $D$  is a PVD and  $U(T_{\text{red}}) = U(K_{\text{red}})$ .

Evidently, ( $\gamma$ ) would follow if we prove both ( $\alpha$ ) and ( $\beta$ ). We next show how to adapt the proof of [15, Theorem 3.1 (i)] in order to prove ( $\alpha$ ) and ( $\beta$ ).

In carrying out the above-mentioned adaptation in order to prove ( $\alpha$ ), we only need to know that some prime ideal of  $D = R/\text{Nil}(R)$  contains a nonzero element (called “ $y$ ” on [15, p. 181, line 9]). This is, in fact, the case since the assumption that  $R$  has more than one prime ideal ensures that  $\text{Nil}(R)$  is not a maximal ideal of  $R$  (cf. [18, Theorem 10]).



As for the proof of  $(\beta)$ , the only possible difficulty in carrying out the above adaptation is to legitimize the meaning of “ $(uv)^{-1}$ ” (as on [15, p. 181, line 14]), where  $Q \in \text{Spec}(\phi(R))$  and  $u, v \in T_{red} \setminus Q$ . This, in turn, is handled by the assumption that  $T$  has only one minimal prime ideal, for  $T_{red}$  is then a domain.

Now that the above claim has been proved, we will show that to use  $(\alpha)$  to prove (a1) and how to use  $(\beta)$  to prove (a2). This will complete the proof of (a), as it is clear that (a3) is just the result of combining (a1) and (a2).

Recall from [6, Proposition 2.9] that  $R$  is a  $\phi$ -PVR if and only if  $D$  is a PVD. Therefore, in view of Theorem 2.1, it suffices to show that the following two conditions are equivalent:

- ( $\lambda$ )  $\text{Nil}(\phi(R)) = \text{Nil}(T)$  and  $U(T_{red}) = U(K_{red})$ ;
- ( $\mu$ )  $U(T) = U(K)$ .

The proof of this equivalence depends on the following well-known key fact: in any (commutative) ring  $A$ , the sum of any unit of  $A$  and any nilpotent element of  $A$  is a unit of  $A$ . This “key fact” easily implies that if  $A$  is any ring, then  $U(A_{red}) = \{a + \text{Nil}(A) \in A/\text{Nil}(A) =: A_{red} \mid a \in U(A)\}$ . In particular,  $U(T_{red}) = \{t + \text{Nil}(T) \mid t \in U(T)\}$  and  $U(K_{red}) = \{k + \text{Nil}(K) \mid k \in U(K)\}$ .

Suppose that  $(\lambda)$  holds. Since  $\phi(R) \subseteq K \subseteq T$ , it follows that  $\text{Nil}(\phi(R)) = \text{Nil}(K) = \text{Nil}(T)$ . To prove  $(\mu)$ , it is enough to show that if  $t \in U(T)$ , then  $t \in K$ . By the “key fact” and  $(\lambda)$ , we have that  $t + \text{Nil}(T) \in U(T_{red}) = U(K_{red})$ , whence  $t + \text{Nil}(T) = t + \text{Nil}(K) = k + \text{Nil}(K)$  for some  $k \in U(K)$ . Thus,  $t \in k + \text{Nil}(K) \subseteq K$ , as desired.

It remains to show that  $(\mu) \Rightarrow (\lambda)$ . Suppose that  $(\mu)$  holds. By the above descriptions of  $U(T_{red})$  and  $U(K_{red})$ , it is enough to prove that  $\text{Nil}(\phi(R)) = \text{Nil}(T)$  (hence necessarily  $= \text{Nil}(K)$ ). Of course,  $\text{Nil}(\phi(R)) \subseteq \text{Nil}(T)$ . For the reverse inclusion, consider any  $w \in \text{Nil}(T)$ . Then, by the “key fact” and  $(\mu)$ , we have that  $1 + w \in U(T) = U(K) \subseteq K$ , whence  $w = (1 + w) - 1 \in K$  and  $w \in \text{Nil}(K) = \text{Nil}(R_{\text{Nil}(R)}) = \text{Nil}(\phi(R))$ , the last step being provided by the fact (iv) that was recalled in the Introduction. This completes the proof of (a).

(b) Note that the intersections mentioned in the statement of (b) make sense by virtue of Lemma 3.1 (c). The proof of (b) proceeds by assuming that  $\phi(R) \subseteq T$  is a strong extension and adapting the proof of [15, Theorem 3.1 (ii)], with changes such as “ $\phi(R)$ ” replacing “ $R$ ” as needed. In particular, by using the fact that  $T \cap R_{\text{Nil}(R)}$  is an overring of  $\phi(R)$ , one obtains a non-zero-divisor  $r$  of  $\phi(R)$  (rather than just a nonzero element) such that  $r(xy) \in \phi(R)$ . With the observation that  $r^{-1} \in \text{Tot}(\phi(R)) = R_{\text{Nil}(R)}$ , the proof is then easily adapted.

(c) Note that the intersections mentioned in the statement of (b) make sense by virtue of Lemma 3.1 (a). The proof of (c) proceeds by adapting the proof of

[15, Theorem 3.1 (iii)], with “ $\phi(R)$ ” replacing “ $R$ ” and “ $R_{Nil(R)}$ ” replacing “ $K$ ” as needed. In particular, one works with a prime ideal  $P$  of  $\phi(R)$  and elements  $x, y \in T$  such that  $xy \in P$ . The only possible difficulty in adapting the proof given in [15] is that “ $x^{-1}$ ” may not be meaningful. However, in that case,  $x$  is an element of the unique maximal ideal of  $R_{Nil(R)}$ ; i.e.,  $x \in Nil(\phi(R)) \subseteq P$ , which is a desirable outcome.

(d) Note that the intersection mentioned in the statement of (d) make sense by virtue of Lemma 3.1 (c). The proof of (d) proceeds by adapting the proof of [15, Theorem 3.1 (iv)], with “ $\phi(R)$ ” replacing “ $R$ ” and “ $R_{Nil(R)}$ ” replacing “ $K$ ” as needed. Our task is to show that if  $x \in U(T)$ , then  $x \in \phi(R) = T \cap R_{Nil(R)}$ , or equivalently, that  $x \in R_{Nil(R)}$ . Because of the hypothesis that  $R$  has more than one prime ideal, Lemma 3.2 (a) yields a prime ideal  $P$  of  $\phi(R)$  containing some non-zero-divisor  $y$ . As in the proof in [15], it is enough to show that if  $yx^{-1} \in P$ , then  $x \in \phi(R)$ , or equivalently, that  $x \in R_{Nil(R)}$ . This can be done by aping the calculation in [15] once one notices that  $w := yx^{-1}$  is a non-zero-divisor in  $\phi(R)$ . (In view of the proof of Lemma 3.1 (a), the point is that  $y$  remains a non-zero-divisor in the ring of fractions  $S := T_{\phi(R) \setminus Z(\phi(R))}$ , whence  $w$  is a non-zero-divisor in  $S$  and, *a fortiori*, a non-zero-divisor in its subring  $\phi(R)$ .) It follows that “ $(yx^{-1})^{-1}$ ” is meaningful in  $\text{Tot}(\phi(R)) = R_{Nil(R)}$  and the proof can now be completed as in [15].  $\square$

We pause to explain the unavoidability of the hypothesis in Theorem 3.3 (a1), (a3) that  $R$  has more than one prime ideal. The underlying reason also accounts for the riding hypothesis in [15, Section 3] that the ambient domain there was not a field. In fact, without these hypotheses, the “only if” conclusions of Theorem 3.3 (a3) and [15, Theorem 3.1 (i)] would each fail. To see this, suppose that  $K \subset T$  is an extension of (distinct) fields. Since 0 is trivially  $T$ -strong, we see that  $\phi_K(K) = K \subseteq T$  is a strong extension. However,  $U(T) = T \setminus \{0\} \neq K \setminus \{0\} = U(K) = U(K_{Nil(K)})$ .

In the spirit of Corollaries 2.12, 2.15 and 2.17, one immediately infers the following special case of Theorem 3.3.

**Corollary 3.4.** *Let  $R \in \mathcal{H}$  such that  $Z(R) = Nil(R)$ , and let  $T$  be any ring that contains  $R$  as a (unital) subring. Then:*

(a) *Assume also that  $(\text{Tot}(R) =) R_{Nil(R)} \subseteq T$ . Then:*

(a1) *If, in addition,  $R$  has more than one prime ideal and  $R \subseteq T$  is a strong extension, then  $R$  is a  $\phi$ -PVR and  $U(T) = U(R_{Nil(R)})$ .*

(a2) *If, in addition,  $T$  has only one minimal prime ideal and both  $R$  is a  $\phi$ -PVR and  $U(T) = U(R_{Nil(R)})$ , then  $R \subseteq T$  is a strong extension.*

(a3) If, in addition,  $R$  has more than one prime ideal and  $T$  has only one minimal prime ideal, then  $R \subseteq T$  is a strong extension if and only if both  $R$  is a  $\phi$ -PVR and  $U(T) = U(R_{Nil(R)})$ .

(b) If  $R \subseteq T$  is a strong extension, then both  $R \subseteq T \cap R_{Nil(R)}$  and  $T \cap R_{Nil(R)} \subseteq T$  are strong extensions.

(c) Assume also that each non-zero-divisor of  $R$  is a non-zero-divisor of  $T$ . If  $R \subseteq T \cap R_{Nil(R)}$  satisfies LO and if both  $R \subseteq T \cap R_{Nil(R)}$  and  $T \cap R_{Nil(R)} \subseteq T$  are strong extensions, then  $R \subseteq T$  is a strong extension.

(d) Assume also that  $R$  has more than one prime ideal and that each non-zero-divisor of  $\phi(R)$  is a non-zero-divisor in  $T$ . If  $R \subseteq T$  is a strong extension such that  $T \cap R_{Nil(R)} = R$ , then  $U(T) = U(R)$ .

In the spirit of parts (b) and (c) of Corollary 3.4, we close by generalizing [15, Proposition 3.3] from domains to rings.

**Proposition 3.5.** *Let  $R \subseteq A \subseteq T$  be rings such that  $A$  is integral over  $R$ . Then  $R \subseteq T$  is a strong extension if and only if both  $R \subseteq A$  and  $A \subseteq T$  are strong extensions.*

PROOF. The “if” assertion is immediate from Proposition 2.18. Conversely, suppose that  $R \subseteq T$  is a strong extension. Then, *a fortiori*,  $R \subseteq A$  is a strong extension. Hence, by Proposition 2.18, these two facts imply that  $\text{Spec}(R) = \text{Spec}(T)$  and  $\text{Spec}(R) = \text{Spec}(A)$ , respectively. As  $\text{Spec}(A) = \text{Spec}(T)$ , another application of Proposition 2.18 shows that  $A \subseteq T$  is a strong extension.  $\square$

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